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# Characterizing the Optimal Profile of the Open Pit Problem in the Continuous Framework

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In this article we propose further properties of the optimal profile for the open pit problem, in the continuous framework, that is to say the case in which the problem is written in terms of a functional space. In particular, we prove that the benefit along the border of the optimal pit is zero, unless the slope or capacity restrictions are active.

## Introduction

In this work we address three problems: the final open pit problem (FOP) in which the optimal profile must only satisfy the slope condition; the capacitated final open pit (CFOP), in which we add a capacity condition, imposing that the total mass to be extracted is limited by a given upper bound; and the capacitated dynamic open pit problem (CDOP), in which a nested sequence of capacitated profiles is determined along the periods. In the two first cases, the criteria is maximizing the total benefit, but in the third case the criteria is given by the discounted value for the planning horizon. We essentially prove that the value distribution along the lower border of the optimal pit must be zero or negative, when the slope or capacity constraint are not actives. This result comes from the tools provided by the calculus of variations and optimal control. Some relevant references on open pit models are [1], [2], [3], [4].

## Preliminaries

Let $\Omega$ be the region of interest in $\mathbb{R}^2$ or $\mathbb{R}^3$, supposed to be open and bounded, and $\overline{\Omega}$ its closure. Each pit will be defined by a (profile) function $p : \overline{\Omega} \to \mathbb{R}$, where $p(x)$ represents the depth of the pit at the point $x \in \Omega$. We suppose that these profiles belong to the Banach space of real continuous functions $\mathcal{C}(\overline{\Omega})$, under the supreme norm. We say that a profile $q$ is deeper than $p$ if $p(x) \leq q(x)\ \forall x \in \overline{\Omega}$. We also consider an initial (feasible) profile $p_0 \in \mathcal{C}(\overline{\Omega})$ and the two regularity conditions

$$p(x) - p_0(x) \geq 0,\ \forall x \in \overline{\Omega}$$

$$p(x) - p_0(x) = 0,\ \forall x \in \partial\Omega$$

Note that we also consider that for $\underline{z} := \inf\left\{p_0(x) | x \in \overline{\Omega}\right\}$ and a given $\overline{z}$, we have $p(x) \in Z = [\underline{z}, \overline{z}],\ \ \forall x \in \overline{\Omega}$. To include the slope condition we use $L_p(x) \leq w(x, p(x)),\ \ \forall x \in \overline{\Omega}$, where

$$L_p(x) := \limsup_{\bar{x} \to x \leftarrow \hat{x}} \frac{|p(\bar{x}) - p(\hat{x})|}{||\bar{x} - \hat{x}||}\ \ \forall x \in \overline{\Omega}$$

and $w : \overline{\Omega} \times Z \to [0, \infty)$ is a given function.
We define the effort and the gain functions: $e(x, z) \geq e_o > 0,\ \ g(x, z) \in \mathbb{R},\ \ \forall (x, z) \in \Omega \times Z$, supposed uniformly bounded, and

$$G[p, q] := \int_\Omega \int_{p(x)}^{q(x)} g(x, z) dz dx, \qquad E[p, q] := \int_\Omega \int_{p(x)}^{q(x)} e(x, z) dz dx$$

$$G[q] := G[p_0, q] \qquad E[q] := E[p_0, q]$$

Then we write the problem

$$(FOP) \quad \max G[p]$$

$$p(x) - p_0(x) \geq 0, \qquad \forall x \in \overline{\Omega} \tag{1}$$

$$p(x) - p_0(x) = 0, \qquad \forall x \in \partial\Omega \tag{2}$$

$$L_p(x) \leq w(x, p(x)) \qquad \forall x \in \overline{\Omega} \tag{3}$$

$$p \in \mathcal{C}(\overline{\Omega}) \tag{4}$$

We denote by $\mathcal{P}$ the set of feasible profiles of problem $(FOP)$.
The capacitated final open pit is (for a given constant $C$)

$$
\begin{aligned}
(CFOP) \quad & \max G[p] & & \\
p(x) - p_0(x) &\geq 0, & \forall x \in \overline{\Omega} \quad & (5)\\
p(x) - p_0(x) &= 0, & \forall x \in \partial\Omega \quad & (6)\\
L_p(x) &\leq w(x, p(x)) & \forall x \in \overline{\Omega} \quad & (7)\\
E[p] &\leq C & & (8)\\
p &\in \mathcal{C}(\overline{\Omega}) & & (9)
\end{aligned}
$$

We recall here three relevant results from [1].

**Proposition 1** *If $w$ is upper semi-continuous, then the set of feasible pits of $(FOP)$ is compact in $\mathcal{C}(\overline{\Omega})$.*

**Proposition 2** *If $g, e \in L^\infty$ then $E$ and $G$ are Lipschitz continuous functions.*

**Proposition 3** *If $w$ is upper semi-continuous and $g, e \in L^\infty$ then*

*i) Problem $(FOP)$ posses at least a solution and there exists a pair of unique profiles $\underline{p}_g$, $\overline{p}_g$ such that $\underline{p}_g \leq p \leq \overline{p}_g$, for every $p$, solution of $(FOP)$.*

*ii) For every capacity $C > 0$, problem $(CFOP)$ posses at least a solution.*

We turn now to the $(CDOP)$ problem, in which we seek to determine a nested sequence of feasible pits having a maximum value of the discounted benefit along the horizon planning $t \in [0, T]$. We denote this sequence by $p(t, \cdot) \in \mathcal{P}, \quad \forall t \in [0, T]$.

Let $c \in L^\infty(0, T), \;\; c(t) \geq 0 \;\; \forall t \in [0, T]$ be the capacity at time $t$ and

$$C(s, t) = \int_s^t c(\tau) d\tau$$

the total capacity in the interval $[s, \, t] \subseteq [0, \, T]$.

Moreover, we define a monotonically decreasing function $\varphi \in \mathcal{C}^1[0, \, T]$, such that $\varphi(0) = 1$ and $0 < \varphi(T) < 1$. Typically we use $\varphi(t) = e^{-\delta t}$, for a given $\delta > 0$.

Then, the problem $(CDOP)$ can be written as:

$$
\begin{aligned}
(CDOP) \quad \max G_D[p] := \int_\Omega \int_0^T \varphi(t) g(x, p(t, x)) \frac{\partial p}{\partial t}(t, x) dt dx & & \\
p(t, x) = p_0(x) \quad & \forall x \in \partial\Omega, \; t \in [0, T] & (10)\\
L_{p(t,\cdot)}(x) \leq w(x, p(t, x)) \quad & \forall x \in \Omega, \; t \in [0, T] & (11)\\
p_0 = p(0, \cdot) \leq p(s, \cdot) \leq p(t, \cdot) \quad & \forall s, t \in [0, \, t], \; s \leq t & (12)\\
\int_\Omega \int_{p(s,x)}^{p(t,x)} e(x, z) dz dx \leq C(s, t) \quad & , \forall s, t \in [0, \, t], \; s \leq t & (13)\\
p(t, \cdot) \in \mathcal{C}^1(\overline{\Omega}) \quad & & (14)
\end{aligned}
$$

The objective function can be written in the following form (integration by parts):

$$G_D[p] := \varphi(T) \int_\Omega \int_{p(0,x)}^{p(T,x)} g(x, z) dz dx + \int_\Omega \int_0^T -\varphi'(t) \int_{p(0,x)}^{p(t,x)} g(x, z) dz dt dx$$

The next proposition permits to guaranty the continuity of the profiles with respect to the time (see [1]).

**Proposition 4** *Let $p(t, x)$ be a feasible point of the previous problem. Then, for every $s, t \in [0, T]$ we have*

$$\|p(t, \cdot) - p(s, \cdot)\|_\infty \leq \left[ \frac{\|c\|_\infty}{\pi e_0} + 2\bar{w} \right] (t - s)^{\frac{1}{3}}$$

## An operational form of the open pit problem

**Proposition 5** *If* $p \in \mathcal{C}^1(\Omega)$ *then* $L_p(x) = ||\nabla p(x)|| \;\; \forall x \in \Omega.$

**Proof.** Let $\bar{x}, \hat{x} \in \Omega$. As $p \in \mathcal{C}^1(\Omega)$ the mean value theorem in $\mathbb{R}^n$ says that there exists $\xi_{\bar{x},\hat{x}} \in co\{\bar{x}, \hat{x}\}$ such that

$$|p(\bar{x} - \hat{x})| \leq ||\nabla p(\xi_{\bar{x},\hat{x}})|| ||\bar{x} - \hat{x}||$$

which is equivalent to

$$\frac{|p(\bar{x} - \hat{x})|}{||\bar{x} - \hat{x}||} \leq ||\nabla p(\xi_{\bar{x},\hat{x}})||$$

By taking $\limsup\limits_{\bar{x} \to x \leftarrow \hat{x}}$ and considering $\limsup\limits_{\bar{x} \to x \leftarrow \hat{x}} \nabla p(\xi_{\bar{x},\hat{x}}) = \nabla p(x)$, because the function $p$ is continuously differentiable, and from the continuity of the norm, we have

$$L_p(x) \leq ||\nabla p(x)||$$

Now, by fixing $\hat{x} = x$ and $\bar{x} \neq x,\; \bar{x} \to x$, then lim sup is equal to $||\nabla p(x)||$ reaching the upper bound and $L_p(x) = ||\nabla p(x)||$. ∎

This proposition permits to rewrite the constraint on $L_p$, when we use the space $\mathcal{C}^1$. This assumption exclude several feasible profiles but the constraint become more operational. In the following section we will work in the functional space of piecewise differentiable functions instead of $\mathcal{C}^1$, in the particular case of a bidimensional mine (i.e., $\Omega \subseteq \mathbb{R}$).

## Optimality conditions for the open pit problem using calculus of variations

We will use write the Fritz-John conditions for a critical point of the open pit problem. Firstly, we recall some standard properties of the following generic problem:

$$(CV) \quad \min F[z] = \int_a^b f(x, z(x), \dot{z}(x))dx$$

$$z(a) = \alpha, \;\; z(b) = \beta$$

$$q_j(x, z(x), \dot{z}(x)) \leq 0 \;\; j = 1, .., m$$

$$x \in [a, b]$$

where the notation $\dot{z}$ represents the derivative with respect to the real variable $x$.

**Definition 6** *A function* $y : [a, b] \to \mathbb{R}^n$ *is piecewise continuous in* $[a, b]$ *if*

- $y$ *is bounded in* $[a, b]$.
- *There exists the left limit of* $y$ *in* $(a, b]$ *and the right limit of* $y$ *in* $[a, b)$.
- $y$ *is continuous in* $[a, b]$*, except in a finite subset of* $[a, b]$.

**Definition 7** *A function* $y : [a, b] \to \mathbb{R}$ *is piecewise differentiable in* $[a, b]$*, if there exists a function* $v$*, piecewise continuous in* $[a, b]$*, such that*

$$y(x) = v(a) + \int_a^x v(s)ds \;\; \forall x \in [a, b]$$

*That is to say,* $y$ *is piecewise differentiable if it is continuous in* $[a, b]$ *and its derivative is piecewise continuous in* $[a, b]$.

We denote by $X = C([a, b], \mathbb{R}^n)$ the functional space containing the functions $y : [a, b] \to \mathbb{R}^n$, piecewise differentiable, equipped with the norm: $\|y\| = \|y\|_\infty + \|Dy\|_\infty$. We denote by $K$ the feasible set of the problem (CV), over the space of piecewise differentiable functions. This space is more general than $\mathcal{C}^1$ and it add the profiles having a finite number of vertexes.

**Definition 8** *We call $z \in K$ a Fritz-John critical point for problem $(CV)$, if there exists the Lagrange multiplier $\tau \in \mathbb{R},\ y = (y_1, .., y_m) : [a, b] \to \mathbb{R}^m$ such that $\forall x \in [a, b]$, excepting on discontinuities, we have:*

$$\tau f_z(x, z, \dot{z}) + y^T(x) q_z(x, z, \dot{z}) = \frac{d}{dx}\left\{\tau f_{\dot{z}}(x, z, \dot{z}) + y^T(x) q_{\dot{z}}(x, z, \dot{z})\right\} \quad (15)$$

$$y^T(x) q(x, z, \dot{z}) = 0 \quad (16)$$

$$(\tau, y(x)) \neq 0,\ \tau \geq 0,\ y_j(x) \geq 0,\ j = 1, .., m \quad (17)$$

Note that if $\tau > 0$ then, if we consider $\tau = 1$, the solution $z$ becomes a Karush-Kuhn-Tucker point. Now, we enounce two relevant theorems from [5], that will be used in our further analysis.

**Theorem 9** *Let $\bar{z} \in K$ be an optimal solution of $(CV)$. Then, $\bar{z}$ is a Fritz-John critical point.*

**Theorem 10** *Let $z \in K$ y $f(\cdot, z(\cdot), \dot{z}(\cdot))$, $q(\cdot, z(\cdot), \dot{z}(\cdot))$ be convex functions in $[a, b]$. If $z$ is a Fritz-John critical point, then $z$ is an optimal solution of $(CV)$.*

The precedent theorems can also be founded in [6] , in more general context.

## FOP and CFOP Problems

In the sequel we will analyze the behavior of the value density along the boundary of the pit in $\mathcal{C}^1$ solution of $(FOP)$ for a $\mathbb{R}^2$ mine. Then we consider the problem:

$$(FOP_o) \qquad \max \int_a^b \int_{p_0(x)}^{p(x)} g(x, z)\, dz\, dx$$

$$p(a) = p_0(a), \quad p(b) = p_0(b) \quad (18)$$

$$p(x) \geq p_0(x) \quad \forall x \in [a, b] \quad (19)$$

$$|\dot{p}(x)| \leq w(x, p(x)) \quad \forall x \in (a, b) \quad (20)$$

$$p \in C([a, b], \mathbb{R}) \quad (21)$$

By identifying this problem with the notation in the previous section, we have:

$$f(x, p, \dot{p}) = -\int_{p_0(x)}^{p(x)} g(x, z)\, dz$$

$$q_1 = p_0(x) - p(x) \leq 0$$

$$q_2 = \dot{p}(x) - w(x, p(x)) \leq 0$$

$$q_3 = -\dot{p}(x) - w(x, p(x)) \leq 0$$

By applying Theorem 9 we obtain the following result:

**Proposition 11** *Let $p \in K$ an optimal profile. If*

- *$g \in \mathcal{C}$*
- *$w(x, z)$ is derivable with respect to $z$*
- *there exists $\bar{a}, \bar{b}$ such that $\quad a < \bar{a} < \bar{b} < b \quad$ and $\quad |\dot{p}(x)| < w(x, p(x)) \quad \forall x \in (\bar{a}, \bar{b})$,*

*then for every $x \in (\bar{a}, \bar{b})$ the following statements hold:*

*(i)* $g(x, p(x)) \leq 0$

*(ii)* $(p(x) - p_0(x)) g(x, p(x)) = 0$

**Proof.** From the precedent theorem, the optimal solution $p$ is a Fritz-John critical point for $(FOP_o)$.
As $g$ is continuous, by the fundamental theorem of calculus we have that

$$f_p(x,p,\dot{p}) = -g(x,p(x))$$

Moreover,

$$(q_1)_p = -1 \qquad\qquad (q_2)_p = (q_3)_p = -w_p(x,p(x))$$

On the other hand, as $f$ and $g_1$ doesn't depend on $\dot{p}$, we have that

$$f_{\dot{p}} = (q_1)_{\dot{p}} = 0, \qquad\qquad (q_2)_{\dot{p}} = 1, \qquad\qquad (q_3)_{\dot{p}} = -1$$

As $p$ is a Fritz-John point, there exist $\tau \in \mathbb{R}$ and $y = (y_1, y_2, y_3) : [a,b] \to \mathbb{R}^3$, piecewise differentiable that such:

$$\begin{aligned}
-\tau g(x,p(x)) - y_1(x) - w_p(x,p(x))(y_2(x) + y_3(x)) &= \frac{d}{dx}\{y_2(x) - y_3(x)\} && (22)\\
y_1(x)(p_0(x) - p(x)) &= 0 && (23)\\
y_2(x)(\dot{p}(x) - w(x,p(x))) &= 0 && (24)\\
y_3(x)(-\dot{p}(x) - w(x,p(x))) &= 0 && (25)\\
(\tau, y(x)) &\neq 0 \quad \tau \geq 0 && (26)\\
y_j(x) &\geq 0 \quad j = 1,..,3 && (27)
\end{aligned}$$

Now we consider the fact that $|\dot{p}(x)| < w(x,p(x)) \ \ \forall x \in ]\bar{a}, \bar{b}[$. From slack conditions, we have $\forall x \in ]\bar{a}, \bar{b}[$, except for discontinuities:

$$y_2(x) = y_3(x) = 0$$

which implies

$$\frac{dy_2(x)}{dx} = \frac{dy_3(x)}{dx} = 0.$$

Replacing in (22) we obtain $\tau g(x,p(x)) = -y_1(x)$.

Note that $\tau = 0$ implies $y_1(x) = 0$ and as we already know, $y_2(x) = y_3(x) = 0$, then $(\tau, y(x)) = 0$. This is a contradiction. Hence, $\tau \neq 0$ and:

$$g(x,p(x)) = -\frac{y_1(x)}{\tau}$$

But $\tau, y_1 \geq 0$, then we obtain $g(x,p(x)) \leq 0$.

On the other hand, by replacing $y_1$ in (23) we have:

$$-\tau g(x,p(x))(p_0(x) - p(x)) = 0,$$

which implies

$$g(x,p(x))(p_0(x) - p(x)) = 0$$

Then $(i)$ and $(ii)$ hold $\ \forall x \in (\bar{a}, \bar{b})$, except for discontinuities. By continuity of $g, p$ and $p_0$ this can be extended to the whole interval. ■

The derivability of $w$ could seems to be a restrictive hypothesis but, in real cases, this is often considered as a constant.

Proposition 11 essentially says that, if the optimal profile in a given point doesn't hit the maximal slope, then the benefit in that point must be zero or negative.

The previous analysis is based on continuity of the benefit function $g$, with respect to the second variable, that is, the vertical coordinate of the mine. This is a strong assumption, because in many real mines the distribution of the richness in the site is quite variable and sometimes with very big jumps in value. But, from a mathematical point of view this assumption permits to derive important properties.

Let us turn now to the $(CFOP)$ problem. The only change is the additional constraint

$$\int_a^b \int_{p_0(x)}^{p(x)} e(x,z)dzdx \leq \overline{E}$$

where $e(x,z) \geq 0$ is the effort function (capacity). So, we define

$$q_4 = \int_a^b \int_{p_0(x)}^{p(x)} e(x,z)dzdx - \overline{E}$$

and the Fritz-John points are given by:

$$-\tau g(x,p(x)) - y_1(x) - w_p(x,p(x))(y_2(x)+y_3(x)) + y_4(x)\int_a^b e(x,p(x))dx = \frac{d}{dx}\{y_2(x) - y_3(x)\} \quad (**)$$

$$y_1(x)(p_0(x) - p(x)) = 0$$

$$y_2(x)(\dot{p}(x) - w(x,p(x))) = 0$$

$$y_3(x)(-\dot{p}(x) - w(x,p(x))) = 0$$

$$y_4(x)\left(\int_a^b \int_{p_0(x)}^{p(x)} e(x,z)dzdz - \overline{E}\right) = 0$$

$$(\tau, y(x)) \neq 0 \quad \tau \geq 0$$

$$y_j(x) \geq 0, \quad j = 1,..,4$$

Then, we can propose the following result for the $(CFOP)$ case.

**Proposition 12** *Let $p \in K$ be an optimal profile for $(CFOP)$. If*

- *$g \in \mathcal{C}$,*
- *$w(x,z)$ is derivable with respect to $z$*
- *there exist $\bar{a}, \bar{b}$, such that $a < \bar{a} < \bar{b} < b$ and $|\dot{p}(x)| < w(x,p(x)) \ \forall x \in ]\bar{a},\bar{b}[$,*

*then $\forall x \in ]\bar{a},\bar{b}[$:*

*(i)* $\left(\overline{E} - \int_a^b \int_{p_0(x)}^{p(x)} e(x,z)dzdz\right) g(x,p(x)) \leq 0$

*(ii)* $(p(x) - p_0(x))g(x,p(x))\left(\int_a^b \int_{p_0(x)}^{p(x)} e(x,z)dzdz - \overline{E}\right) = 0$

This result can be easily interpreted: if the optimal profile doesn't use the maximal slope, one possibility is that the gain function is negative in the zone or because the maximal capacity is completely fulfilled.

## CDOP Problem

In the dynamic problem introduced in [1] the authors consider a continuous time interval $[0,T]$ and, for every time $t$ they seek for a feasible profile for $(FOP)$ in a decreasing sequence along the time. To write this new problem they introduce $p(t,x)$, which represents the depth in the point $x$ at time $t$, which means that $p(t,\cdot)$ is a feasible profile for $(FOP)$.

As we showed before, this problem can be formulated in the following form.

$$(CDOP) \quad \max G[p] := \varphi(T)\int_a^b \int_{p(0,x)}^{p(T,x)} g(x,z)dzdx + \int_a^b \int_0^T -\varphi'(t)\int_{p(0,x)}^{p(t,x)} g(x,z)dzdtdx$$

$$p(t,a) = p_0(a), \;\; p(t,b) = p_0(b) \;\forall t \in [0,T] \tag{28}$$

$$|\dot{p}(t,x)| \leq w(x,p(t,x)) \quad \forall x \in (a,b) \;\forall t \in [0,T] \tag{29}$$

$$p_0 = p(0,\cdot) \leq p(s,\cdot) \leq p(t,\cdot), \;\; \forall\, 0 \leq s \leq t \leq T \tag{30}$$

$$\int_a^b \int_{p(s,x)}^{p(t,x)} e(x,z)dzdx \leq \int_s^t c(\tau)d\tau, \;\; \forall\, 0 \leq s \leq t \leq T \tag{31}$$

$$p(t,\cdot) \in C([a,b],\mathbb{R}) \tag{32}$$

**Proposition 13** *Let $p^N(T,\cdot) \in K$ the profile at time $T$ (discrete time in $N$ intervals). If*

- $g \in \mathcal{C}$
- *$w(x,z)$ is derivable with respect to $z$*
- *existen $\bar{a}, \bar{b}$, such that $a < \bar{a} < \bar{b} < b$ y $|\dot{p}(x)| < w(x,p(x))$ $\forall x \in ]\bar{a},\bar{b}[$*

*then $\forall x \in ]\bar{a},\bar{b}[$:*

*(i)* $\left(\int_{t_{N-1}}^T c(\tau)d\tau - \int_a^b \int_{p^N(t_{N-1},x)}^{p^N(T,x)} e(x,z)dzdx\right) g(x,p^N(T,x)) \leq 0$

*(ii)* $(p^N(T,x) - p^N(t_{N-1},x))\left(\int_{t_{N-1}}^T c(\tau)d\tau - \int_a^b \int_{p^N(t_{N-1},x)}^{p^N(T,x)} e(x,z)dzdx\right) g(x,p^N(T,x)) = 0$

Now we replicate the previous results, for the special case of continuous space and time. From now, $p(t,x)$ $t \in [0,T]$, $x \in [a,b]$ represents a solution of $(CDOP)$, and we define the family of problems, $\forall \epsilon \in ]0,T[$

$$(P_\epsilon) \quad \max \int_a^b \int_{p(T-\epsilon,x)}^{p(x)} \varphi(T)g(x,z)dzdx$$

$$p(a) = p(T-\epsilon,a), \;\; p(b) = p(T-\epsilon,b) \tag{33}$$

$$|\dot{p}(x)| \leq w(x,p(x)) \quad \forall x \in (a,b) \tag{34}$$

$$p(T-\epsilon,x) \leq p(x) \quad \forall x \in [a,b] \tag{35}$$

$$\int_a^b \int_{p(T-\epsilon,x)}^{p(x)} e(x,z)dzdx \leq \int_{T-\epsilon}^T c(\tau)d\tau \tag{36}$$

$$p \in C([a,b],\mathbb{R}) \tag{37}$$

This problem have the form of $(CFOP)$, with initial profile $p(T-\epsilon)$ and capacity $\int_{T-\epsilon}^T c(\tau)d\tau$. The solution of this problem is denoted by $p_\epsilon$.

**Proposition 14** *Let $p(T,\cdot)$ be the profile, at time $T$, of the solution of $(CDOP)$. If there exists a sequence $\epsilon_k \in ]0,T[$, $\epsilon_k \xrightarrow{k\to\infty} 0$ such that $(P_{\epsilon_k})$ posses a solution $p_{\epsilon_k}$, then*

$$p_{\epsilon_k}(x) \xrightarrow{k\to\infty} p(T,x) \;\; a.e.$$

**Proposition 15** *Let $p(T,\cdot) \in K$ be the profile at time $T$ for Problem $(CDOP)$. Also, let us suppose that $g \in \mathcal{C}$ and $w(x,z)$ is derivable with respect to $z$. If there exists a sequence $\{\epsilon_k\} \subset [0,T]$, converging to zero, such that $(P_{\epsilon_k})$ possess solution and there exists $\bar{k} \in \mathbb{N}$, $\bar{a}, \bar{b}$, $a < \bar{a} < \bar{b} < b$ such that $\forall k \geq \bar{k}$:*

- $|\dot{p}_{\epsilon_k}| < w(x, p_{\epsilon_k}(x)) \quad \forall x \in ]\bar{a}, \bar{b}[$
- $\displaystyle\int_a^b \int_{p(T-\epsilon_k,x)}^{p_{\epsilon_k}(x)} e(x,z)dzdx < \int_{T-\epsilon_k}^{T} c(\tau)d\tau$

*then* $\forall x \in ]\bar{a}, \bar{b}[$

$$g(x, p(T,x)) \leq 0 \;\; a.e.$$

*Moreover, if* $p(T-\epsilon_k, x) < p_{\epsilon_k}(x)$ *then* $g(x, p(T,x)) = 0$ *a.e.*

This continuous version can be formulated as a optimality control problem whit state constraints where prepositions 11 and 12 still true. The content for this purpose is presented in [7], [8].

## Conclusion

The problems $(P_\epsilon)$ can be interpreted as a maximization of the benefit at time $T-\epsilon$, without consider the past, under a given capacity $\displaystyle\int_{T-\epsilon}^{T} c(\tau)d\tau$. We have essentially prove that under non active capacity or slope constraint, the lower border of the pit takes the value zero or negative. This could provide good suggestions for pit optimization algorithms, by seeking for final or sequenced pits satisfying this condition. This work is a beginning to generalize previous proposition to $\mathbb{R}^3$ mine.


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