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Analytical Properties of the Feasible and Optimal Profiles in the Binary Programming Formulation of Open Pit

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In open pit mine planning, there are three well known models concerning how to find a region of maximal economic value under geotechnical stability constraints. These models belong to the area of integer programming and we respectively call them as Final Open Pit (FOP), Capacitated Open Pit (COP) and Capacitated Dynamic Open Pit (CDOP). In this work we formalize some ideas about these models and their connecting properties and, as a main result, we prove the well known conjecture that a pit which is solution of (CDOP) is contained in the biggest pit from the solution of (FOP).

Introduction

Three models are usually considered in this area, such as is pointed out in [1], [2] and [3]. We will describe the main properties of this models aiming to provide a new proof of the well known fact that the sequenced open pit is contained in the final open pit. The case with ore blending constraints or lower bounds in the capacity are not considered in this work and certainly our main result could change in those cases. To formalize this approach, we first recall the mathematical definitions of those problems.
Let us suppose that we have a set of blocks $\mathcal{N}$ which the benefit is known, then for every $i \in \mathcal{N}$ we call b_i that value, and define the variables:

$$x_i = \begin{cases} 1 \text{ if block } i \text{ is chosen for extraction} \\ 0 \text{ if not} \end{cases}$$

We also consider a set of arcs $\mathcal{A}$ representing the precedences between the bloks. The optimization model can be expressed by

$$(FOP) \quad \max \sum_{i \in N} b_i x_i$$
$$x_j - x_i \geq 0 \quad \forall (i,j) \in A \tag{1}$$
$$x_i \in \{0,1\} \quad \forall i \in N \tag{2}$$

If we consider a maximal capacity C of extraction in terms of total mass, the model can be modified as

$$(CFOP) \quad \max \sum_{i \in N} b_i x_i$$
$$x_j - x_i \geq 0 \quad \forall (i,j) \in A \tag{3}$$
$$\sum_{i \in N} p_i x_i \leq C \tag{4}$$
$$x_i \in \{0,1\} \quad \forall i \in N \tag{5}$$

p_i being the density of the block. In this case, if the total capacity is expressed in terms of number of blocks, then we simply use $p_i = 1$, for all $i \in \mathcal{N}$ and C is an integer number.
For the dynamic problem, we define the variables:

$$x_i^t = \begin{cases} 1 & \text{if block } i \text{ is extracted at time t} \\ 0 & \text{if not} \end{cases}$$

In this case, we consider discrete time $t = 1, \ldots, T$ and a constant $\alpha \in]0,1[$ representing the discount factor. We also know the capacity at every period, denoted as C_t. The dynamic model can be written as:

$$(CDOP) \quad \max \sum_{t=1}^{T} \sum_{i \in N} \frac{b_i}{(1+\alpha)^{t-1}} x_i^t$$

$$\sum_{t=1}^{T} x_i^t \leq 1 \quad \forall i \in N \tag{6}$$

$$\sum_{l=1}^{t} x_j^l - x_i^t \geq 0 \quad \forall (i,j) \in A, \ t = 1, ..., T \tag{7}$$

$$\sum_{i \in N} p_i x_i^t \leq C_t \quad t = 1, ..., T \tag{8}$$

$$x_i^t \in \{0, 1\} \quad \forall i \in N, t = 1, ..., T \tag{9}$$

In [3] the constraint (8) has a lower bound, which in this case is 0.

Definition 1 *For every feasible point x we will denote $P_x = \{i \in \mathcal{N} \mid x_i = 1\}$ the pit associated to x. If x is a solution of (FOP), we call P_x a final (optimal) pit.*

Definition 2 *For every y, feasible point of $(CDOP)$, we denote $Q_y = \{i \in \mathcal{N} \mid \exists t \in \{1, ..., T\} \ y_i^t = 1\}$ the associated pit to y and $Q_y^t = \{i \in N | y_i^t = 1\}$ is known as a phase. Obviously,*

$$Q_y = \bigcup_{t=1}^{T} Q_y^t$$

The solution of (FOP) is in general not unique, but we consider the solution having maximal cardinality. See, as example, the case:

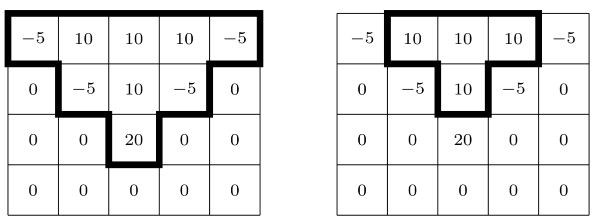

Figure 1: Two solutions of the same (FOP) problem

The right pit has the same value as the left pit, however the latter contains more blocks. The conjecture is:

Conjecture 3 *Let $\bar{x}$ be the solution of (FOP) such that $|P_{\bar{x}}|$ is maximal among all optimal solutions of (FOP) and let y be any optimal solution of $(CDOP)$, then:*

$$Q_y \subseteq P_{\bar{x}}$$

Later, we will prove that $P_{\bar{x}}$ contains all the pits having maximal value. Moreover, it is clear that if a final pit contains all the pits having maximal value then has maximal cardinality, so our conjecture can be presented as follows.

Conjecture 4 *Let $\bar{x}$ a solution of (FOP) such that, every optimal solution is contained in $P_{\bar{x}}$, and let y be any optimal solution of $(CDOP)$, then:*

$$Q_y \subseteq P_{\bar{x}}$$

Both conjectures are equivalent so we talk about the conjecture. The existence of this optimal pit having maximal cardinality comes from the fact that the feasible set is finite and nonempty (the trivial solution is always feasible).

Preliminary results

Firstly, we show some preliminary results to prove the conjecture.

Proposition 5 *If x^1 and x^2 are feasible points of (FOP) then there exists a feasible point z such that:*

$$P_{x^1} \cup P_{x^2} = P_z$$

Proof: It is clear that

$$P_{x^1} \cup P_{x^2} = \left\{i \in \mathcal{N} | x_i^1 = 1\right\} \cup \left\{i \in \mathcal{N} | x_i^2 = 1\right\} = \left\{i \in \mathcal{N} | x_i^1 = 1 \vee x_i^2 = 1\right\}$$

Then, we have to prove that there exists a feasible point z such that:

$$\left\{i \in \mathcal{N} | x_i^1 = 1 \vee x_i^2 = 1\right\} = \{i \in \mathcal{N} | z_i = 1\}$$

By defining the binary variable z such that $z_i = 1 \iff x_i^1 = 1 \vee x_i^2 = 1$ we clearly have the equality of the sets. To verify (1), we analyze two cases

a) $z_i = 0$. In this case $z_j - z_i = z_j \geq 0 \; \forall (i,j) \in \mathcal{A}$ and the constraint is satisfied.

b) $z_i = 1$. By definition of z, this means that $x_i^1 = 1 \vee x_i^2 = 1$. Given that these two vectors are feasible, from the precedence relations we have:

$$x_j^1 = 1 \vee x_j^2 = 1 \; \forall (i,j) \in \mathcal{A}$$

This implies $z_j = 1 \; \forall (i,j) \in \mathcal{A}$, then $z_j - z_i \geq 0 \; \forall (i,j) \in \mathcal{A}$. ■

Proposition 6 *If x^1 and x^2 are feasible for (FOP) then there exists z, feasible point for (FOP), such that:*

$$P_{x^1} \cap P_{x^2} = P_z$$

Proof. Following the same ideas of the previous proposition, we have

$$P_{x^1} \cap P_{x^2} = \left\{i \in N \,|\, x_i^1 = 1 \wedge x_i^2 = 1\right\}$$

Then, if we define the binary variable z such that $z_i = 1 \iff x_i^1 = 1 \wedge x_i^2 = 1$ we have the equality of the two sets. The relevant case to prove (1) is $z_i = 1$, because if $z_i = 0$ the constraint is trivially satisfied.
The expression $z_i = 1$ implies $x_i^1 = 1$ and $x_i^2 = 1$, then given that x^1 and x^2 are feasible for (FOP), we have $x_j^1 = x_j^2 = 1 \; \forall (i,j) \in \mathcal{A}$.
So $z_j = 1 \; \forall (i,j) \in \mathcal{A}$, then z satisfies $P_{x^1} \cap P_{x^2} = P_z$ and is feasible for (FOP). ■

Remark 7 *From now, given a feasible point x for (FOP), we will use the following expression for the objective function:*

$$\sum_{i \in N} b_i x_i = \sum_{i \in N : x_i = 1} b_i = \sum_{i \in P_x} b_i$$

Proposition 8 *If x is any solution of (FOP), and $\bar{x}$ a solution of (FOP) which $|P_{\bar{x}}|$ is the biggest among all solutions of (FOP) then $P_x \subseteq P_{\bar{x}}$*

Proof. From Proposition 5, $P_x \cup P_{\bar{x}}$ is well defined in a feasible point of (FOP), then

$$\sum_{i \in P_{\bar{x}}} b_i \geq \sum_{i \in P_x \cup P_{\bar{x}}} b_i = \sum_{i \in P_{\bar{x}}} b_i + \sum_{i \in P_x \setminus P_{\bar{x}}} b_i \tag{10}$$

The first inequality comes from the optimality of $\bar{x}$ and the equality comes from the fact that $P_{\bar{x}}$ and $P_x \setminus P_{\bar{x}}$ is a partition of $P_x \cup P_{\bar{x}}$. Then we have:

$$\sum_{i \in P_x \setminus P_{\bar{x}}} b_i \leq 0 \tag{11}$$

Similarly,

$$\sum_{i\in P_x} b_i = \sum_{i\in P_x\cap P_{\bar{x}}} b_i + \sum_{i\in P_x\setminus P_{\bar{x}}} b_i \leq \sum_{i\in P_x\cap P_{\bar{x}}} b_i + 0$$

Now, as x and $\bar{x}$ are feasible points for (FOP), from Proposition 6 there exists a feasible point for (FOP) such that the associated final pit is $P_x \cap P_{\bar{x}}$, then the last inequality must be an equality (because x is optimal for (FOP)). Then

$$\sum_{i\in P_x\setminus P_{\bar{x}}} b_i = 0$$

By replacing in (10) we have

$$\sum_{i\in P_x\cup P_{\bar{x}}} b_i = \sum_{i\in P_{\bar{x}}} b_i$$

Then, from the existence of z, feasible for (FOP) such that $P_x \cup P_{\bar{x}} = P_z$, then it is also an optimum.
From the definition of $\bar{x}$ as the solution of (FOP) such that $P_{\bar{x}}$ has maximal cardinality then

$$|P_x \cup P_{\bar{x}}| \leq |P_{\bar{x}}|$$

But $P_{\bar{x}} \subseteq P_x \cup P_{\bar{x}}$, then the equality holds. So

$$|P_{\bar{x}}| + |P_x \setminus P_{\bar{x}}| = |P_x \cup P_{\bar{x}}| = |P_{\bar{x}}| \implies |P_x \setminus P_{\bar{x}}| = 0$$

which means $P_x \setminus P_{\bar{x}} = \emptyset$ and $P_x \subseteq P_{\bar{x}}$. ■

Corollary 9 *There is only one solution of (FOP) whose associated pit has the biggest cardinal among all the solutions of (FOP)*

Proof. The existence is because of the feasibility of the problem. Let x^1 and x^2 be two optimum points of (FOP) whose associated pit has the biggest cardinal. Because of preposition 2.3 we have that

$$P_{x^1} \subseteq P_{x^2} \wedge P_{x^2} \subseteq P_{x^1}$$

So $P_{x^1} = P_{x^2}$ concluding that $x^1 = x^2$.

Main results

In the sequel, we refer to the **unique** solution of (FOP) having biggest cardinal among all the solutions of (FOP).

Proposition 10 *Given a feasible point y for $(CDOP)$, there exists z, feasible point for (FOP), such that*

$$Q_y = P_z$$

Proof. Let $y = \{y_i^t\}$ be a feasible point of $(CDOP)$. We define

$$z_i = \sum_{t=1}^{T} y_i^t$$

We will prove that z is feasible for (FOP) and it fulfill the equality of the corresponding sets of blocks. Firstly, from feasibility of y it is obvious that $z_i \in \{0,1\}$, because of (6) and (9).
To prove $z_j - z_i \geq 0\ \forall(i,j) \in A$ we only consider the case $z_i = 1$, the other one being trivial. It is clear that there exists a unique $t = 1,\ldots,T$ such that $y_i^{\hat{t}} = 1$. From (7) we have, for all $(i,j) \in \mathcal{A}$

$$1 = y_i^{\hat{t}} \leq \sum_{l=1}^{\hat{t}} y_j^l \leq \sum_{l=1}^{T} y_j^l = z_j$$

which means that $z_j = 1\ \forall(i,j) \in \mathcal{A}$ and then z is feasible for (FOP).

Furthermore, it is easy to see that

$$z_i = 1 \iff \sum_{t=1}^{T} y_i^t = 1 \iff \exists t \in \{1,...,T\}\ y_i^t = 1$$

where the last equivalence is due to feasibility of y for (CDOP). This implies $Q_y = P_z$. ■

We prove below some new properties of the optimal pits.

Proposition 11 *Let $\bar{y}$ be any solution of (CDOP) and $\bar{x}$ be the solution whit maximal cardinality of (FOP). Then*

$$\sum_{i \in Q_{\bar{y}} \setminus P_{\bar{x}}} b_i = 0 \implies Q_{\bar{y}} \subseteq P_{\bar{x}}$$

Proof. From Proposition 10 there exist $\bar{z}$ such that $Q_{\bar{y}} = P_{\bar{z}}$ and from Proposition 5, $P_{\bar{z}} \cup P_{\bar{x}}$ contains a feasible point for (FOP). Furthermore

$$\sum_{i \in P_{\bar{z}} \cup P_{\bar{x}}} b_i = \sum_{i \in P_{\bar{x}}} b_i + \sum_{i \in P_{\bar{z}} \setminus P_{\bar{x}}} b_i = \sum_{i \in P_{\bar{x}}} b_i,$$

then the vector determining $P_{\bar{z}} \cup P_{\bar{x}}$ is a solution of (FOP). From Proposition 8 we have $P_{\bar{z}} \cup P_{\bar{x}} \subseteq P_{\bar{x}}$ which implies $P_{\bar{z}} \subseteq P_{\bar{x}}$. ■

Later, we are going to prove this last preposition is necessary so the conjecture is right.

Proposition 12 *Let y be a feasible point of $(CDOP)$, x a feasible point of (FOP) and $\bar{t} \in \{1, \ldots, T-1\}$. Then there exists w, feasible point for $(CDOP)$ such that*

(i) $Q_w^t = Q_y^t\ for\ t \in \{1,...,\bar{t}\},\ \bar{t} \geq 1$

(ii) $Q_w^t = Q_y^t \cap P_x\ for\ t \in \{\bar{t}+1,...T\}$

And when $\bar{t} = 0$ there exist w feasible point for (CDOP) such that

$$Q_w^t = Q_y^t \cap P_x\ for\ t \in \{1,...T\}$$

Proof. For $\bar{t} \geq 1$ let us define w by

$$w_i^t = \begin{cases} y_i^t & \text{si } t \in \{1,...,\bar{t}\},\ i \in N \\ 1 & \text{si } (y_i^t = 1\ \wedge\ x_i = 1),\ t \in \{\bar{t}+1,...,T\},\ i \in N \\ 0 & \text{si } (y_i^t = 0\ \vee x_i = 0),\ t \in \{\bar{t}+1,...,T\},\ i \in N \end{cases}$$

Then it is clear that

$$Q_w^t = Q_y^t\ for\ t \in \{1,...,\bar{t}\},\ \bar{t} \geq 1$$
$$Q_w^t = Q_y^t \cap P_x\ for\ t \in \{\bar{t}+1,...T\},$$

We will prove that this w is a feasible point for $(CDOP)$. Note that by how we define w, $w_i \in \{0,1\}$ and this can be written as $w_i^t = y_i^t x_i$ for $t \in \{\bar{t}+1,...T\}$, which implies that $w_i^t \leq y_i^t\ \forall t \in \{1,...,T\}$, then

$$\sum_{t=1}^{T} w_i^t \leq \sum_{t=1}^{T} y_i^t \leq 1\ \forall i \in N$$

$$\sum_{i \in N} p_i w_i^t \leq \sum_{i \in N} p_i y_i^t \leq C_t\ \forall t \in \{1,...T\}$$

This prove (6),(7) and (9). For the precedence constraints we have that $y = w$ when $t \in \{1,...,\bar{t}\}$, then the constraint is trivially satisfied. The interesting case to analyze is $t \in \{\bar{t}+1,...,T\}$. If $w_i^t = 0$ the precedence constraint $\sum_{l=1}^{t} w_j^l - w_i^t = \sum_{l=1}^{t} w_j^l \geq 0$ is trivially satisfied $\forall (i,j) \in \mathcal{A}$.
Now, let $w_i^t = 1$. By definition of w, we have $y_i^t = 1$ and $x_i = 1$ and let us consider j such that que $(i,j) \in \mathcal{A}$. Given that y is feasible point for $(CDOP)$, there exists $l \in \{1,...,t\}$ such that $y_j^l = 1$.

- If $l \leq \bar{t}$ then $y_j^l = w_j^l = 1$ which proves the condition.
- If $l > \bar{t}$ we have $y_j^l = 1$ and x being a feasible point for (FOP), we have $x_j = 1$, then $w_j^l = 1$, and the constraint is satisfied.

When $\bar{t} = 0$, we have that $w_i^t = y_i^t x_i,\ ,t \in \{\bar{t}+1, ..., T\}, \forall i \in N.$ The verification of w satisfies (6), (7), (8), (9) can be made by using similar arguments. ■

Theorem 13 Fundamental Theorem. *Let $\bar{x}$ the solution of (FOP) such that the set $P_{\bar{x}}$ has maximal cardinality among all solutions of (FOP), and let y be any solution of $(CDOP)$. Then*

$$Q_y \subseteq P_{\bar{x}}$$

Proof. We are going to prove the hypothesis of Proposition 11. For this, we will prove $\sum_{i \in Q_y \setminus P_{\bar{x}}} b_i = 0$.
From Propositions 5 and 10, we have that $P_{\bar{x}} \cup Q_y$ is a feasible solution of (FOP), then

$$\sum_{i \in P_{\bar{x}} \cup Q_y} b_i = \sum_{i \in P_{\bar{x}}} b_i + \sum_{i \in Q_y \setminus P_{\bar{x}}} b_i \leq \sum_{i \in P_{\bar{x}}} b_i$$

So, we have $\sum_{i \in Q_y \setminus P_{\bar{x}}} b_i \leq 0$
Concerning the other inequality we prove the next result: $\forall \bar{t} \in \{0, ..., T-1\}$

$$\sum_{t=\bar{t}+1}^{T} \sum_{i \in Q_y^t \setminus P_x} b_i \geq \sum_{t=\bar{t}+1}^{T} \sum_{i \in Q_y^t \setminus P_x} \frac{b_i}{(1+\alpha)^{t-(\bar{t}+1)}} \geq 0$$

This will be proven by reverse induction, starting from $T-1$.
Base case: $\bar{t} = T-1$.
From Proposition 12, there exists a vector w, feasible for $(CDOP)$ such that:

$$Q_w^t = Q_y^t \ for\ t \in \{1, ..., T-1\}$$

$$Q_w^t = Q_y^t \cap P_x \ for\ t = T$$

From optimality of y we have:

$$\begin{aligned}
\sum_{t=1}^{T} \sum_{i \in Q_y^t} \frac{b_i}{(1+\alpha)^{t-1}} &= \sum_{t=1}^{T-1} \sum_{i \in Q_y^t} \frac{b_i}{(1+\alpha)^{t-1}} + \sum_{i \in Q_y^T \cap P_x} \frac{b_i}{(1+\alpha)^{T-1}} + \sum_{i \in Q_y^T \setminus P_x} \frac{b_i}{(1+\alpha)^{T-1}} \\
&\geq \sum_{t=1}^{T} \sum_{i \in Q_w^t} \frac{b_i}{(1+\alpha)^{t-1}} \\
&= \sum_{t=1}^{T-1} \sum_{i \in Q_y^t} \frac{b_i}{(1+\alpha)^{t-1}} + \sum_{i \in Q_y^T \cap P_x} \frac{b_i}{(1+\alpha)^{T-1}}
\end{aligned}$$

Then,

$$\sum_{i \in Q_y^T \setminus P_x} \frac{b_i}{(1+\alpha)^{T-1}} \geq 0$$

Due to $(1+\alpha)^{T-1} \geq 0$, so

$$\sum_{i \in Q_y^T \setminus P_x} b_i \geq 0$$

Furthermore,

$$\sum_{t=T}^{T} \sum_{i \in Q_y^t \setminus P_x} b_i = \sum_{t=T}^{T} \sum_{i \in Q_y^t \setminus P_x} \frac{b_i}{(1+\alpha)^{t-T}}$$

which verifies the base case.
Induction: Let us suppose true for $0 < \bar{t} < T-1$. We will prove for $\bar{t}-1,\ \ t \geq 2$.
From Proposition 12 there exists a feasible point w such that

$$Q_w^t = Q_y^t \text{ for } t \in \{1, ..., \bar{t}-1\}$$

$$Q_w^t = Q_y^t \cap P_x \text{ for } t \in \{\bar{t}, ...T\}$$

By using the optimality of y we have

$$\begin{aligned}
\sum_{t=1}^{T} \sum_{i \in Q_y^t} \frac{b_i}{(1+\alpha)^{t-1}} &= \sum_{t=1}^{\bar{t}-1} \sum_{i \in Q_y^t} \frac{b_i}{(1+\alpha)^{t-1}} + \sum_{t=\bar{t}}^{T} \sum_{i \in Q_y^t \cap P_x} \frac{b_i}{(1+\alpha)^{t-1}} + \sum_{t=\bar{t}}^{T} \sum_{i \in Q_y^t \setminus P_x} \frac{b_i}{(1+\alpha)^{t-1}} \\
&\geq \sum_{t=1}^{T} \sum_{i \in Q_w^t} \frac{b_i}{(1+\alpha)^{t-1}} \\
&= \sum_{t=1}^{\bar{t}-1} \sum_{i \in Q_y^t} \frac{b_i}{(1+\alpha)^{t-1}} + \sum_{t=\bar{t}}^{T} \sum_{i \in Q_y^t \cap P_x} \frac{b_i}{(1+\alpha)^{t-1}}
\end{aligned}$$

Then, we have:

$$\sum_{t=\bar{t}}^{T} \sum_{i \in Q_y^t \setminus P_x} \frac{b_i}{(1+\alpha)^{t-1}} \geq 0$$

Given that $(1+\alpha)^{\bar{t}-1} \geq 0$, multiplying this expression we obtain

$$\sum_{t=\bar{t}}^{T} \sum_{i \in Q_y^t \setminus P_x} \frac{b_i}{(1+\alpha)^{t-\bar{t}}} \geq 0$$

For the second inequality we note:

$$\begin{aligned}
0 \leq \sum_{t=\bar{t}}^{T} \sum_{i \in Q_y^t \setminus P_x} \frac{b_i}{(1+\alpha)^{t-\bar{t}}} &= \sum_{i \in Q_y^{\bar{t}} \setminus P_x} b_i + \sum_{t=\bar{t}+1}^{T} \sum_{i \in Q_y^t \setminus P_x} \frac{b_i}{(1+\alpha)^{t-\bar{t}}} \\
&= \sum_{i \in Q_y^{\bar{t}} \setminus P_x} b_i + \frac{1}{(1+\alpha)} \sum_{t=\bar{t}+1}^{T} \sum_{i \in Q_y^t \setminus P_x} \frac{b_i}{(1+\alpha)^{t-(\bar{t}+1)}}
\end{aligned}$$

By using the first inequality of the induction hypothesis

$$0 \leq \sum_{t=\bar{t}+1}^{T} \sum_{i \in Q_y^t \setminus P_x} \frac{b_i}{(1+\alpha)^{t-(\bar{t}+1)}}$$

and, from $\frac{1}{(1+\alpha)} \leq 1$, we obtain:

$$\frac{1}{(1+\alpha)} \sum_{t=\bar{t}+1}^{T} \sum_{i \in Q_y^t \setminus P_x} \frac{b_i}{(1+\alpha)^{t-(\bar{t}+1)}} \leq \sum_{t=\bar{t}+1}^{T} \sum_{i \in Q_y^t \setminus P_x} \frac{b_i}{(1+\alpha)^{t-(\bar{t}+1)}}$$

By using the second inequality of the induction hypothesis

$$\sum_{t=\bar{t}+1}^{T} \sum_{i \in Q_y^t \setminus P_x} \frac{b_i}{(1+\alpha)^{t-(\bar{t}+1)}} \leq \sum_{t=\bar{t}+1}^{T} \sum_{i \in Q_y^t \setminus P_x} b_i$$

Combining the two expressions, we obtain

$$\frac{1}{(1+\alpha)}\sum_{t=\bar{t}+1}^{T}\sum_{i\in Q_y^t\setminus P_x}\frac{b_i}{(1+\alpha)^{t-(\bar{t}+1)}}\leq\sum_{t=\bar{t}+1}^{T}\sum_{i\in Q_y^t\setminus P_x}b_i$$

Then,

$$0\leq\sum_{i\in Q_y^{\bar{t}}\setminus P_x}b_i+\frac{1}{(1+\alpha)}\sum_{t=\bar{t}+1}^{T}\sum_{i\in Q_y^t\setminus P_x}\frac{b_i}{(1+\alpha)^{t-(\bar{t}+1)}}\leq\sum_{i\in Q_y^{\bar{t}}\setminus P_x}b_i+\sum_{t=\bar{t}+1}^{T}\sum_{i\in Q_y^t\setminus P_x}$$

Now, for $\bar{t}=1$, using prop. 12 in the case 0, there exist w such that: $Q_w^t=Q_y^t\cap P_x\ para\ t\in\{1,...T\}$
Developing the expression

$$\begin{aligned}\sum_{t=1}^{T}\sum_{i\in Q_y^t}\frac{b_i}{(1+\alpha)^{t-1}}&=\sum_{t=1}^{T}\sum_{i\in Q_y^t\cap P_x}\frac{b_i}{(1+\alpha)^{t-1}}+\sum_{t=1}^{T}\sum_{i\in Q_y^t\setminus P_x}\frac{b_i}{(1+\alpha)^{t-1}}\\&\geq\sum_{t=1}^{T}\sum_{i\in Q_w^t}\frac{b_i}{(1+\alpha)^{t-1}}\\&=\sum_{t=1}^{T}\sum_{i\in Q_y^t\cap P_x}\frac{b_i}{(1+\alpha)^{t-1}}\end{aligned}$$

Then we have

$$\sum_{t=1}^{T}\sum_{i\in Q_y^t\setminus P_x}\frac{b_i}{(1+\alpha)^{t-1}}\geq 0$$

The proof for the other inequality is similar to the previous proof, so in this case we have that:

$$0\leq\sum_{t=1}^{T}\sum_{i\in Q_y^t\setminus P_x}b_i=\sum_{i\in Q_y\setminus P_x}b_i$$

Finally, Proposition 11 implies $Q_y\subseteq P_{\bar{x}}$ ■

In [4] CDOP is formulated differently but equivalent to the presented in this paper so the Fundamental Theorem also is right in this case.

Conclusions

The Fundamental Theorem is a very known property, but its demonstration is less known. From now on, this paper is a good source to see its proof. The importance of this theorem is that it lets us reduce the quantity of variables and restriction of the CDOP problem, solving first the FOP problem, which is faster and whose solution contains the CDOP solution.

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References

[1] Caccetta, L. Application of optimisation techniques in open pit mining. In: Handbook of Operations Research in Natural Resources, pp. 547-559, Springer Verlag, 2007.

[2] Hustrulid, W.A., M. Kuchta and R.K. Martin. Open Pit Mine Planning and Design. Taylor and Francis, 2013.

[3] Newman, A.M., E. Rubio, R. Caro, A. Weintraub and K. Eurek. A review of operations research in mine planning. Interfaces, Vol 40(3), pp. 222-245, 2010.

[4] Amaya, J., D. Espinoza, M. Goycoolea, E. Moreno, Th. Prévost and E. Rubio. A scalable approach to optimal block scheduling. In: Proceedings APCOM, Applications of Computers on Mining Industry, pp. 567-575, 2009.